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Fractional Heisenberg Equation

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Abstract

Fractional derivative can be defined as a fractional power of derivative. The commutator $(i/\hbar)[H,\ .\ ]$, which is used in the Heisenberg equation, is a derivation on a set of observables. A derivation is a map that satisfies the Leibnitz rule. In this paper, we consider a fractional derivative on a set of quantum observables as a fractional power of the commutator $(i/\hbar)[H,\ .\ ]$. As a result, we obtain a fractional generalization of the Heisenberg equation. The fractional Heisenberg equation is exactly solved for the Hamiltonians of free particle and harmonic oscillator. The suggested Heisenberg equation generalize a notion of quantum Hamiltonian systems to describe quantum dissipative processes.

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1 Introduction

The fractional calculus has a long history from 1695, when the derivative of order 1/2 has been described by Leibniz [1, 2]. The theory of derivatives and integrals of non-integer order goes back to Leibniz, Liouville, Grunwald, Letnikov and Riemann. There are many books about fractional calculus and fractional differential equations [1, 2, 3, 4]. Derivatives of fractional order, and fractional differential equations have found many applications in recent studies in physics (see for example [5, 6, 7, 8] and references therein).

In the quantum kinematics, the observables are given by self-adjoint operators. The dynamical description of system is given by a superoperator, which is a rule that assigns to each operator exactly one operator. The natural description of the motion is in terms of the infinitesimal change of the system. The equation of motion for quantum observable is called the Heisenberg equation. For Hamiltonian systems, the infinitesimal superoperator of the Heisenberg equation is defined by some form of derivation.

Fractional derivative can be defined as a fractional power of derivative (see for example [9]). It is known that the infinitesimal generator $(i/\hbar)[H, \ . \ ]$, which is used in the Heisenberg equation, is a derivation of observables. A derivation is a linear map D, which satisfies the Leibniz rule $D(AB) = (DA)B + A(DB)$ for all operators A and B. In this paper, we consider a fractional derivative on a set of observables as a fractional power of derivative. As a result, we obtain a fractional generalization of the Heisenberg equation. It allows us to generalize a notion of quantum Hamiltonian systems. Note that fractional generalization of classical Hamiltonian systems has been suggested in [10]. The suggested fractional Heisenberg equation is exactly solved for the Hamiltonians of free particle and harmonic oscillator. A quantum system that is presented by fractional Heisenberg equation can be considered as a dissipative system [11]. Fractional derivatives can be used as a possible approach to describe an interaction between the system and an environment. Note that it is possible to consider quantum dynamics with low-level fractionality by some generalization of method suggested in [12] (see also [13]).

In Section 2, the fractional power of derivative and the fractional Heisenberg (FH) equation are suggested. In Section 3, the properties of time evolution, which is described by the fractional equation, are considered. In Section 4, the FH equation for free particle is solved. In Section 5. the solution of FH equation with harmonic oscillator Hamiltonian is derived.

2 Fractional derivative and Heisenberg equation

Let us consider a set of quantum observables in the Heisenberg picture. A superoperator $\mathcal{L}$ is a rule that assigns to each operator A exactly one operator $\mathcal{L}(A)$. (About superoperator formalism see for example [14, 15, 16] and references therein.) For Hamiltonian H, let L^-_H be the superoperator given by

$$L^-_H A = \frac{1}{i\hbar}(HA - AH).$$

The operator differential equation

$$\frac{d}{dt}A_t = -L^-_H A_t \tag{1}$$

is called the Heisenberg equation for Hamiltonian systems. The time evolution of a Hamiltonian system is induced by the Hamiltonian H.

It is interesting to obtain a fractional generalization of equation (1). We will consider here concept of fractional power for L^-_H. If L^-_H is a closed linear superoperator with an everywhere dense domain $D(L^-_H)$, having a resolvent $R(z, L^-_H) = (zL_I - L^-_H)^{-1}$ on the negative half-axis, then there exists [19, 20, 21] the superoperator

$$-(L^-_H)^\alpha = \frac{\sin \pi\alpha}{\pi} \int_0^\infty dz\, z^{\alpha-1} R(-z, L^-_H)\, L^-_H \tag{2}$$

defined on $D(L^-_H)$ for $0 < \alpha < 1$. The superoperator $(L^-_H)^\alpha$ is a *fractional power of the Lie left superoperator*.

As a result, we obtain the equation

$$\frac{d}{dt}A_t = -(L^-_H)^\alpha A_t, \tag{3}$$

where t and $H/\hbar$ are dimensionless variables. This is the *fractional Heisenberg equation*.

Note that this equation cannot be presented in the form

$$\frac{d}{dt}A_t = -L^-_{H_{new}} A_t = \frac{i}{\hbar}[H_{new}, A_t] \tag{4}$$

with some operator H_{new}. Therefore, quantum systems described by (3) are not Hamiltonian systems. The systems will be called the fractional Hamiltonian systems (FHS). Usual Hamiltonian systems can be considered as a special case of FHS.

If we consider the Cauchy problem for equation (1) in which the initial condition is given at the time $t = 0$ by A_0, then its solution can be written in the form

$$A_t = \Phi_t A_0.$$

The one-parameter superoperators Φ_t, $t \geq 0$ have the properties

$$\Phi_t \Phi_s = \Phi_{t+s}, \quad (t, s > 0), \quad \Phi_0 = L_I,$$

where L_I is unit superoperator ($L_I A = A$). Then the set of Φ_t, $t \geq 0$, is called the semi-group. Then the superoperator L^-_H is called the generating superoperator, or infinitesimal generator, of the semi-group $\{\Phi_t, t \geq 0\}$.

Let us consider the Cauchy problem for fractional Heisenberg equation (3) in which the initial condition is given by A_0. Then its solution can be presented in the form

$$A_t(\alpha) = \Phi^{(\alpha)}_t A_0,$$

where the superoperators $\Phi^{(\alpha)}_t$, $t > 0$, form a semi-group which will be called the *fractional semi-group*. The superoperator $(L^-_H)^\alpha$ is infinitesimal generator of the semi-group $\{\Phi^{(\alpha)}_t, t \geq 0\}$.

3 Properties of fractional time evolution

Let us consider some properties of time evolution described by a fractional semi-group $\{\Phi^{(\alpha)}_t, t \geq 0\}$.

(1) The superoperators $\Phi^{(\alpha)}_t$ can be constructed in terms of the superoperators Φ_t by the Bochner-Phillips formula [17, 18, 20]

$$\Phi^{(\alpha)}_t = \int_0^\infty ds f_\alpha(t, s) \Phi_s, \quad (t > 0), \tag{5}$$

where $f_\alpha(t, s)$ is defined by

$$f_\alpha(t, s) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+\imath\infty} dz \, \exp(sz - tz^\alpha), \tag{6}$$

where $a, t > 0$, $s \geq 0$, and $0 < \alpha < 1$. The branch of z^α is so taken that $Re(z^\alpha) > 0$ for $Re(z) > 0$. This branch is a one-valued function in the z-plane cut along the negative real axis. The convergence of this integral is obviously in virtue of the convergence factor $\exp(-tz^\alpha)$. By denoting the path of integration in (6) to the union of two paths $r \exp(-i\theta)$, and $r \exp(+i\theta)$, where $r \in (0, \infty)$, and $\pi/2 \leq \theta \leq \pi$, we obtain

$$f_\alpha(t, s) = \frac{1}{\pi} \int_0^\infty dr \, \exp(sr \cos\theta - tr^\alpha \cos(\alpha\theta)) \cdot$$

$$\cdot \sin(sr \sin\theta - tr^\alpha \sin(\alpha\theta) + \theta). \tag{7}$$

If we have a solution A_t of the Heisenberg equation (1), then formula (5) gives the solution

$$A_t(\alpha) = \int_0^\infty ds f_\alpha(t, s) A_s, \quad (t > 0) \tag{8}$$

of fractional equation (3).

(2) In quantum theory, the most important is the class of real superoperators. Let A^* be an adjoint of A. A quantum observable is a self-adjoint operator. If Φ_t is a real superoperator and A is a self-adjoint operator $A^* = A$, then the operator $A_t = \Phi_t A$ is self-adjoint, i.e.,

$$(\Phi_t A)^* = \Phi_t A.$$

A superoperator as a map from a set of observables into itself should be real. All possible dynamics, i.e., temporal evolutions of quantum observables, should be described by real superoperators. Therefore the following statement is very important. If Φ_t is a real superoperator, then $\Phi_t^{(\alpha)}$ is real. It can be proved by using the Bochner-Phillips formula and equation (7).

(3) An adjoint superoperator of Φ_t is a superoperator $\bar{\Phi}_t$, such that

$$Tr[(\bar{\Phi}_t A)^* B] = Tr[A^* \Phi_t(B)].$$

Let us give the basic statement regarding the adjoint superoperator. If $\bar{\Phi}_t$ is an adjoint superoperator of Φ_t, then the superoperator

$$\bar{\Phi}_t^{(\alpha)} = \int_0^\infty ds f_\alpha(t,s) \bar{\Phi}_s, \quad (t > 0),$$

is an adjoint superoperator of $\Phi_t^{(\alpha)}$. We prove this statement by using the Bochner-Phillips formula:

$$Tr[(\bar{\Phi}_t^{(\alpha)} A)^* B] = \int_0^\infty ds f_\alpha(t,s) Tr[(\bar{\Phi}_s A)^* B] =$$

$$= \int_0^\infty ds f_\alpha(t,s) Tr[A^* \Phi_s(B)] = Tr[A^* \Phi_t^{(\alpha)}(B)].$$

Let $\{\bar{\Phi}_t, t > 0\}$ be a semi-group, such that the density matrix operator $\rho_t = \bar{\Phi}_t \rho_0$ is described by the von Neumann equation

$$\frac{d}{dt} \rho_t = \frac{1}{i\hbar} [H, \rho_t].$$

Then the semi-group $\{\bar{\Phi}_t^{(\alpha)}, t > 0\}$ describes the evolution of the density operator

$$\rho_t(\alpha) = \bar{\Phi}_t^{(\alpha)} \rho_0$$

by the fractional equation

$$\frac{d}{dt} \rho_t = -(-L_H^-)^\alpha \rho_t.$$

This is the *fractional von Neumann equation.*

(4) Let Φ_t, $t > 0$, be a positive one-parameter superoperator, i.e., $\Phi_t A \geq 0$ for $A \geq 0$. Using the Bochner-Phillips formula and the property $f_\alpha(t,s) \geq 0$, $(s > 0)$, it is easy to prove that the superoperators $\Phi_t^{(\alpha)}$ are also positive, i.e., $\Phi_t^{(\alpha)} A \geq 0$ for $A \geq 0$.

(5) It is known that $\bar{\Phi}_t$ is a real superoperator if Φ_t is real. Analogously, if $\Phi_t^{(\alpha)}$ is a real superoperator, then $\bar{\Phi}_t^{(\alpha)}$ is real.

4 Fractional free particle

Let us consider the Hamiltonian $H = P^2/2m$, where P is dimensionless variable, and m^{-1} has the action dimension. Then the Heisenberg equation (1) describes a free one-dimensional particle. For $A = Q$, and $A = P$, equation (1) gives

$$\frac{d}{dt} Q_t = \frac{1}{m} P_t, \quad \frac{d}{dt} P_t = 0.$$

The well-known solutions of these equations are

$$Q_t = Q_0 + \frac{t}{m} P_0, \quad P_t = P_0. \tag{9}$$

Using these solutions and the Bochner-Phillips formula, we will obtain the solutions of fractional Heisenberg equations

$$\frac{d}{dt} Q_t = -\frac{1}{m^\alpha} (L^-_{P^2})^\alpha Q_t, \quad \frac{d}{dt} P_t = 0. \tag{10}$$

Note that $(L^-_{P^2})^\alpha \neq L^-_{P^{2\alpha}}$. The solutions of fractional equations (10) have the forms

$$Q_t(\alpha) = \Phi_t^{(\alpha)} Q_0 = \int_0^\infty ds f_\alpha(t,s) Q_s, \quad P_t(\alpha) = P_0,$$

where Q_s is given by (9). Then

$$Q_t = Q_0 + \frac{1}{m} g_\alpha(t) P_0, \quad P_t = P_0,$$

where

$$g_\alpha(t) = \int_0^\infty ds f_\alpha(t,s)\, s.$$

If $\alpha = 1/2$, then we have

$$g_{1/2}(t) = \frac{t}{2\sqrt{\pi}} \int_0^\infty ds\, \frac{1}{\sqrt{s}} e^{-t^2/4s} = \frac{t^2}{2}.$$

Then

$$Q_t = Q_0 - \frac{t^2}{2m} P_0, \quad P_t = P_0. \tag{11}$$

These equations describe a fractional free motion for $\alpha = 1/2$. For the operators (11), the average values and dispersions have the form

$$< Q_t >= x_0 - \frac{t^2}{2m} p_0, \quad < P_t >= p_0,$$

and

$$D_P(t) = \frac{\hbar^2}{2b^2}, \quad D_Q(t) = \frac{b^2}{2}\Big(1 + \frac{\hbar^2 t^4}{m^2 b^4}\Big).$$

Here we use the coordinate representation and the pure state

$$\Psi(x) =< x|\Psi >= (b\sqrt{\pi})^{-1/2} \exp\Big(-\frac{(x-x_0)^2}{2b} + \frac{i}{\hbar} p_0 x\Big). \tag{12}$$

The average value and dispersion are defined by the well-known equations

$$< A_t >= Tr[|\Psi >< \Psi| A_t] =< \Psi|A_t|\Psi >,$$

$$D_A(t) =< A_t^2 > - < A_t >^2 =< \Psi|A_t^2|\Psi > - < \Psi|A_t|\Psi >^2 .$$

5 Fractional Heisenberg equation for harmonic oscillator

Let us consider the Hamiltonian

$$H = \frac{1}{2m} P^2 + \frac{m\omega^2}{2} Q^2, \tag{13}$$

where t and P are dimensionless variables. Then equation (1) describes a harmonic oscillator. For $A = Q$, and $A = P$, equation (1) gives

$$\frac{d}{dt} Q_t = \frac{1}{m} P_t, \quad \frac{d}{dt} P_t = -m\omega^2 Q_t.$$

The well-known solutions of these equations are

$$Q_t = Q_0 \cos(\omega t) + \frac{1}{m\omega} P_0 \sin(\omega t),$$

$$P_t = P_0 \cos(\omega t) - m\omega Q_0 \sin(\omega t). \tag{14}$$

Using these solutions and the Bochner-Phillips formula, we will obtain the solutions of fractional Heisenberg equations

$$\frac{d}{dt}Q_t = -(L_H^-)^\alpha Q_t, \quad \frac{d}{dt}P_t = -(L_H^-)^\alpha P_t, \tag{15}$$

where H is defined by (13). The solutions of fractional equations (15) have the forms

$$Q_t(\alpha) = \Phi_t^{(\alpha)} Q_0 = \int_0^\infty ds f_\alpha(t,s) Q_s,$$

$$P_t(\alpha) = \Phi_t^{(\alpha)} P_0 = \int_0^\infty ds f_\alpha(t,s) P_s. \tag{16}$$

Substitution of (14) into (16) gives

$$Q_t = Q_0 C_\alpha(t) + \frac{1}{m\omega} P_0 S_\alpha(t), \tag{17}$$

$$P_t = P_0 C_\alpha(t) - m\omega Q_0 S_\alpha(t), \tag{18}$$

where

$$C_\alpha(t) = \int_0^\infty ds\, f_\alpha(t,s)\, \cos(\omega s),$$

$$S_\alpha(t) = \int_0^\infty ds\, f_\alpha(t,s)\, \sin(\omega s).$$

Equations (17) and (18) describe solutions of the fractional Heisenberg equations (15) for quantum harmonic oscillator.

If $\alpha = 1/2$, then

$$C_{1/2}(t) = \frac{t}{2\sqrt{\pi}} \int_0^\infty ds\, \frac{\cos(\omega s)}{s^{3/2}}\, e^{-t^2/4s},$$

$$S_{1/2}(t) = \frac{t}{2\sqrt{\pi}} \int_0^\infty ds\, \frac{\sin(\omega s)}{s^{3/2}}\, e^{-t^2/4s}.$$

These functions can be presented through the Macdonald function (see [22], Sec. 2.5.37.1.) such that

$$C_{1/2}(t) = \Big(\frac{\omega t^2}{4\pi}\Big)^{1/4} \Big[e^{+\pi i/8} K_{-1/2}\Big(2e^{+\pi i/4}\sqrt{\frac{\omega t^2}{4}}\Big) + e^{-\pi i/8} K_{-1/2}\Big(2e^{-\pi i/4}\sqrt{\frac{\omega t^2}{4}}\Big)\Big],$$

$$S_{1/2}(t) = i\Big(\frac{\omega t^2}{4\pi}\Big)^{1/4} \Big[e^{+\pi i/8} K_{-1/2}\Big(2e^{+\pi i/4}\sqrt{\frac{\omega t^2}{4}}\Big) - e^{-\pi i/8} K_{-1/2}\Big(2e^{-\pi i/4}\sqrt{\frac{\omega t^2}{4}}\Big)\Big], \tag{19}$$

where $\omega > 0$, and $K_\alpha(z)$ is the Macdonald function [1, 2], which is also called the modified Bessel function of the third kind.

Using (12), we get the average values

$$< Q_t >= x_0 C_\alpha(t) + \frac{1}{m\omega} p_0 S_\alpha(t),$$

$$< P_t >= p_0 C_\alpha(t) - m\omega x_0 S_\alpha(t),$$

and the dispersions

$$D_P(t) = \frac{\hbar^2}{2b^2} C_\alpha^2(t) + \frac{b^2 m^2 \omega^2}{2} S_\alpha^2(t),$$

$$D_Q(t) = \frac{b^2}{2} C_\alpha^2(t) + \frac{\hbar^2}{2b^2 m^2 \omega^2} S_\alpha^2(t).$$

As a result, the fractional harmonic oscillator is a simple dissipative systems. The dispersion of the wave packet is defined by these equations. The solutions are characterized by the fractional damping of the fractional harmonic oscillator. The dumping is described by the modified Bessel function (19) of the third kind. An important property of the evolution described by the fractional equations are that the solutions have power-like tails.

6 Conclusion

In this paper, we consider derivatives of noninteger order as fractional powers of derivative. We derive a fractional generalization of the Heisenberg equation and a generalization of quantum Hamiltonian system to describe open quantum systems. A quantum system that is presented by fractional Heisenberg equation can be considered as a dissipative system. Fractional derivatives can be used as an approach to describe an interaction between the quantum system and an environment. This interpretation caused by following reasons. Using the properties

$$\int_0^\infty f_\alpha(t,s) = 1, \qquad f_\alpha(t,s) \geq 0 \quad (for\ all \quad s > 0),$$

we can assume that $f_\alpha(t,s)$ is a density of probability distribution. Then the Bochner-Phillips formula (5) can be considered as a smoothing of Hamiltonian evolution Φ_t with respect to time $s > 0$. This smoothing can be considered as an influence of the environment on the system. As a result, the parameter alpha can be used as a simple model of the interaction between the system and the environment. Note that the quantum Markovian equations, which is also called the Lindblad equations [23], can be generalized by suggested approach to describe completely positive evolution of dissipative and open quantum systems.

References

[1] K.B. Oldham, J. Spanier, *The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order* (Academic Press, New York, 1974).

[2] S.G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional Integrals and Derivatives Theory and Applications* (Gordon and Breach, New York, 1993).

[3] I. Podlubny, *Fractional Differential Equations* (Academic Press, San Diego, 1999).

[4] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Application of Fractional Differential Equations* (Elsevier, Amsterdam, 2006).

[5] G.M. Zaslavsky, *Hamiltonian Chaos and Fractional Dynamics* (Oxford University Press, Oxford, 2005).

[6] A. Carpinteri, F. Mainardi, (Eds), *Fractals and Fractional Calculus in Continuum Mechanics* (Springer, Wien, 1997).

[7] G.M. Zaslavsky, "Chaos, fractional kinetics, and anomalous transport" Phys. Rep. **371** (2002) 461-580.

[8] E.W. Montroll, M.F. Shlesinger, "The wonderful world of random walks" In: *Studies in Statistical Mechanics*, Vol. 11. J. Lebowitz, E. Montroll (Eds.), (North-Holland, Amsterdam, 1984) pp.1-121.

[9] V.E. Tarasov, "Fractional derivative as fractional power of derivative" Int. J. Math. **18**(3) (2007) 281-299.

[10] V.E. Tarasov, "Fractional generalization of gradient and Hamiltonian systems" J. Phys. A **38** (2005) 5929-5943.

[11] U. Weiss, *Quantum Dissipative Systems* (World Scientific, Singapore, 1993).

[12] V.E. Tarasov, G.M. Zaslavsky, "Dynamics with low-level fractionality" Physica A **368** (2006) 399-415.

[13] A. Tofighi, H.N. Pour, "Epsilon-expansion and the fractional oscillator" Physica A **374** (2007) 41-45.

[14] V.E. Tarasov, "Path integral for quantum operations" J. Phys. A **37** (2004) 3241-3257.

[15] V.E. Tarasov, "Pure stationary states of open quantum systems" Phys. Rev. E **66** (2002) 056116.

[16] V.E. Tarasov, "Quantization of non-Hamiltonian and dissipative systems" Phys. Lett. A **288** (2001) 173-182.

[17] S. Bochner, "Diffusion equations and stochastic processes" Proc. Nat. Acad. Sci USA **35** (1949) 369-370.

[18] R.S. Phillips, "On the generation of semi-groups of linear operators" Pacific J. Math. **2** (1952) 343-396.

[19] V. Balakrishnan, "Fractional power of closed operator and the semi-group generated by them" Pacific J. Math. **10** (1960) 419-437.

[20] K. Yosida, *Functional analysis* (Springer, Berlin, 1965).

[21] S.G. Krein, *Linear Differential Equations in Banach Space*, Translations of Mathematical Monographs, Vol.29, Amer. Math. Soc., 1971.

[22] A.P. Prudnikov, Yu.A. Brychkov, O.I. Marichev, *Elementary Functions, Integrals and Series* Vol.1. (Gordon and Breach, New York, 1986).

[23] G. Lindblad, "On the generators of quantum dynamical semi-groups", Commum. Math. Phys. **48** (1976) 119-130.